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Multi-Period Modeling of Greeks Using Least Squares Monte Carlo: An Exotic Option Case Study

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Overview

Our recent paper "Proxy Functions for the Projection of Variable Annuity Greeks" outlined a methodology for statistically efficient and accurate estimation of the multi-period behavior of VA Greeks through the stochastically-projected run-off of the business. This will be useful for VA businesses that wish to evaluate the likely performance of hedging strategies and demonstrate hedge effectiveness to regulators.

The case study in that paper highlighted the effectiveness of the methodology, but only considered the case of a vanilla put option. This short note extends the analysis of the previous case study by considering a more complex example: a lookback option. We show that the methodology can produce a similar quality of fitting performance for the lookback option as in the vanilla option case. We also discuss the methodology adjustments that are necessary for the Greeks fitting strategy in order to accurately fit to forms of path-dependent, exotic options such as lookbacks.

Contents

1. Introduction

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Section 2 describes the option used in the case study and its assumed valuation model.

Section 3 develops the Greeks fitting methodology, and in particular highlights where it needs to be adjusted from the fitting strategy described in the previous paper.

Section 4 provides out-of-sample validation test results and section 5 concludes.

2. Example liability and valuation model

In this note we consider a specific problem of estimating market-consistent value, Delta, Rho and Vega for a *10-year lookback put option* on an equity index. At maturity T, the option pays the difference between the highest value of the index over the life of the option (S_T^{Max}) and the index value at maturity (S_T):

$$Payoff = S_T^{Max} - S_T$$

To estimate the market-consistent value of this liability, we assume the equity index follows a Geometric Brownian Motion (Black-Scholes model) under the risk-neutral measure. While such a model may appear unrealistically simple, such models are often used to value VA guarantees in practice. Furthermore, for the particular liability considered here, market-consistent value can be calculated using standard analytical formulae while Greeks can be calculated accurately using numerical differentiation of the value function. In the current exercise, such analytical values are useful as they provide a benchmark against which we can validate fitted proxy functions.

3. Fitting methodology

Our first consideration is to identify the risk factors that the option value (and its Greeks) depend on. Given our choice of Black-Scholes valuation model, the value of the lookback option (and hence its Greeks) at some future time t is a function of four risk factors:

» The risk-free interest rate R_t

» The volatility σ_t

» The current equity index S_t

» The maximum value of the index achieved to date S_t^{Max}

For notational convenience we collect the risk factors together as a vector $\underline{RF_t} = (R_t, \sigma_t, S_t, S_t^{Max})$. Our aim is to develop proxy functions for value $V_t^{proxy}(\underline{RF_t})$, delta $\Delta_t^{proxy}(\underline{RF_t})$, rho $\rho_t^{proxy}(\underline{RF_t})$ and vega $\mathcal{V}_t^{proxy}(\underline{RF_t})$ by fitting the function to estimated values at a selection of 'fitting points' $\underline{RF_{i,t}} = (R_{i,t}, \sigma_{i,t}, S_{i,t}, S_{i,t}^{Max})$.

Following on from our experience in developing Greeks proxy functions for vanilla options (Clayton, et al. 2013) we have made the following methodology choices in carrying out these fits:

1. At each fitting point, we estimate each Greek using a 'bump-and-revalue' method, and then fit a proxy function to these estimates. Our previous paper indicates that this method provides superior fits to a potential alternative in which we estimate market-consistent value at each fitting point, fit a proxy function to this, and differentiate this function with respect to the relevant risk factor in order to calculate its Greeks.

   In this paper, the bump-and-revalue estimate at fitting point $\underline{RF_{i,t}} = (R_{i,t}, \sigma_{i,t}, S_{i,t}, S_{i,t}^{Max})$ was defined by bumping the relevant risk factor by a small amount δx (while leaving all other risk factors unchanged) to create a new fitting point, revaluing the option and estimating the derivative as a finite difference. For example, to estimate delta, we create the new fitting point $\underline{RF_{i,t}^{S\,bumped}} = (R_{i,t}, \sigma_{i,t}, S_{i,t} + \delta x, S_{i,t}^{Max})$ and define:

$$\widehat{\Delta}_{i,t} = \frac{\hat{V}\left(\underline{RF_{i,t}^{S\,bumped}}\right) - \hat{V}\left(\underline{RF_{i,t}}\right)}{\delta x}$$

   where $\hat{V}$ is the estimated value, estimated using risk-neutral Monte Carlo simulation.

   In our previous paper, we estimated Greeks using bumped scenarios in which a number of risk factors are bumped simultaneously (rather than stressing individual risk factors independently). This 'cluster' method enabled us to recycle fitting scenarios at different time–steps and thus cut down on the total number of fitting scenarios required. In the current context, we were unable to produce satisfactory fits using this method, due to the strong dependency between the index S_t and the maximum to date S_t^{Max}. As a result we chose to separately bump each individual risk factor as described above.

2. Proxy functions are fitted using a 'local regression' technique (sometimes referred to as LOESS), whereby we fit local polynomial functions at each point in the risk factor space, using a weighted least squares technique with more weight on placed on 'nearby' fitting points (Cleveland and Devlin 1988). For example, if we want to estimate the option delta at time t, we evaluate using a polynomial Δ_t^{proxy} whose coefficients have been chosen so as to minimize the *weighted* sum of squared residuals:

$$\sum_i w\left(\frac{D\left(\underline{RF}_{i,t}, \underline{RF}_t\right)}{h}\right)\left(\widehat{\Delta}_{i,t} - \Delta_t^{proxy}\left(\underline{RF}_t\right)\right)^2$$

where $\widehat{\Delta}_{i,t}$ is the estimated delta in scenario i. $D\left(\underline{RF}_{i,t}, \underline{RF}_t\right)$ describes the 'distance' between the evaluation point $\underline{RF}_t$ and fitting point $\underline{RF}_{i,t}$, with the weight function $w\left(\frac{D\left(\underline{RF}_{i,t}, \underline{RF}_t\right)}{h}\right)$ typically chosen so as to put less weight on fitting points that are further away from the evaluation point.[1] Since the fitting space is adjusted dynamically to reflect a small neighborhood around the chosen point, we can accurately estimate the function in that neighborhood using only low-order polynomials, for example linear functions or even constants.

4. Out-of-Sample Validation Testing

In the specific case of a lookback put option, we have chosen to fit and validate proxy functions at years 1, 5 and 9. We assumed a total fitting scenario budget of 100,000 scenarios per point in time. This consists of 25,000 'base' scenarios and three sets of 25,000 of 'bump' scenarios (corresponding to the three different Greeks). At each time step of interest, base fitting points were sampled to cover a wide range of plausible values, as follows:

» Risk-free interest-rates and volatilities were sampled uniformly, over a range tailored to reflect the range of plausible values at the time step of interest. For example, at year 5, the risk-free rate was sampled in the range [1%,9%] and volatility sampled in the range [15%,35%].

» Conditional on a particular risk-free rate and volatility sample, the equity index was sampled assuming a uniform distribution covering the 1%-99% quantiles as implied by the corresponding Black-Scholes model.

» Conditional on a particular volatility and equity index sample, the maximum index achieved to date was sampled assuming a uniform distribution covering the 1%-99% quantiles as implied by the corresponding Black-Scholes model.

At each time step, an additional 200 points were chosen as out-of-sample validation points. At each validation point, 'actual' market-consistent values were calculated using standard analytical formulae while Greeks were calculated accurately using numerical differentiation of the analytical value.

[1] h is a 'bandwidth' parameter which controls how quickly the weight decays as a function of the distance i.e. how localised the weighting is.

Figures 1, 2 and 3 compare proxy vs. actual values for all three methods at times 1, 5 and 9 years, respectively.

Figure 1: Quality of fit at year 1: proxy vs. actual in 200 validation scenarios across the fitting space

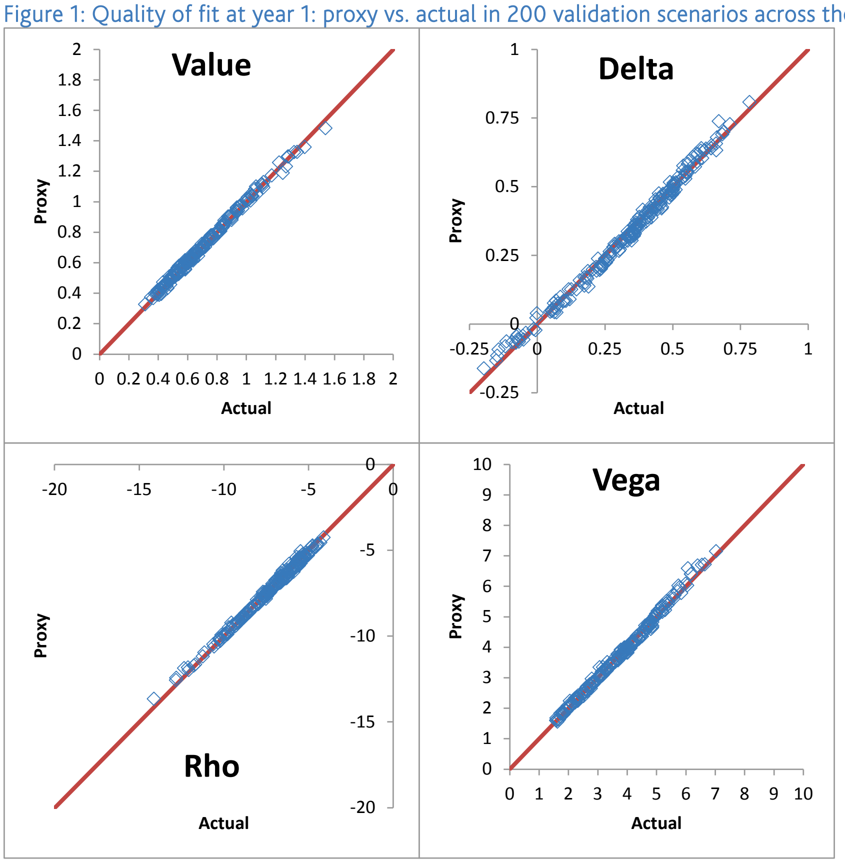


Figure 2: Quality of fit at year 5: proxy vs. actual in 200 validation scenarios across the fitting space

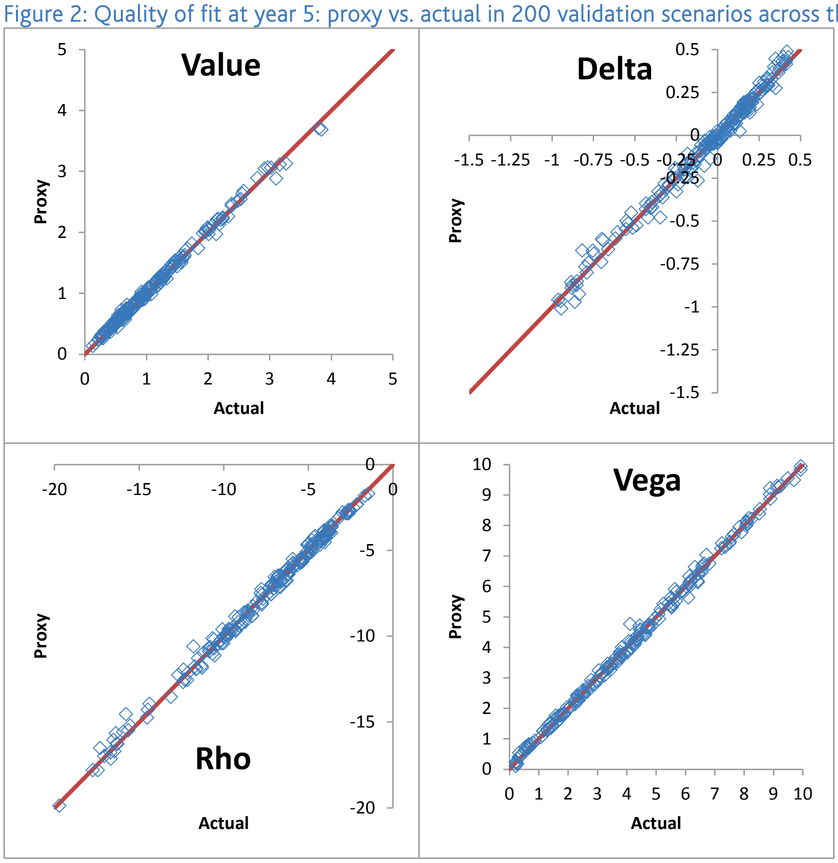
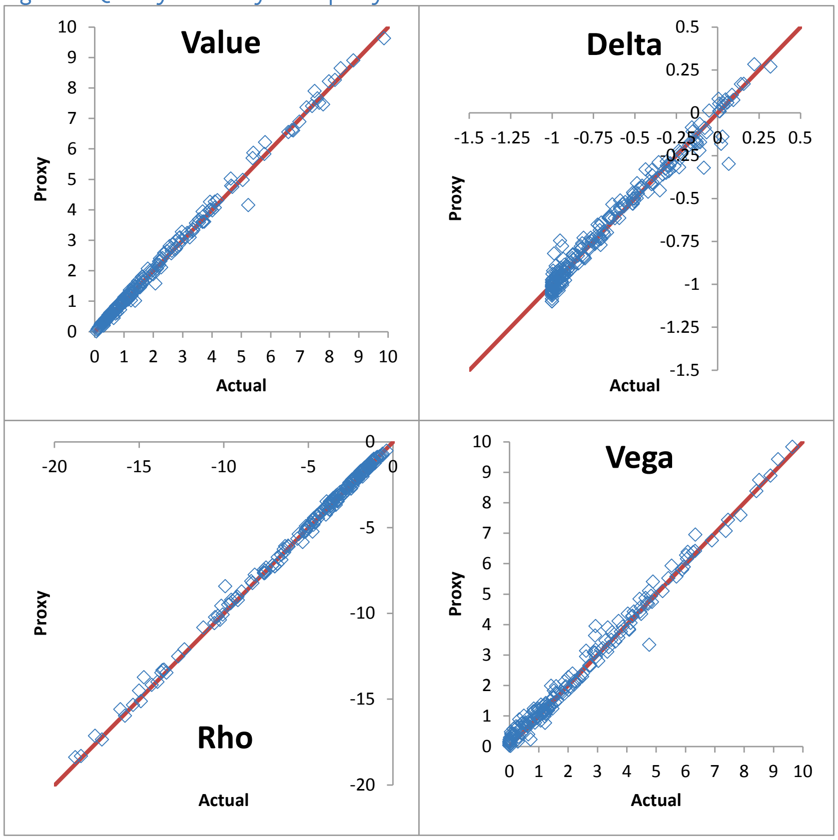


Figure 3: Quality of fit at year 9: proxy vs. actual in 200 validation scenarios across the fitting space

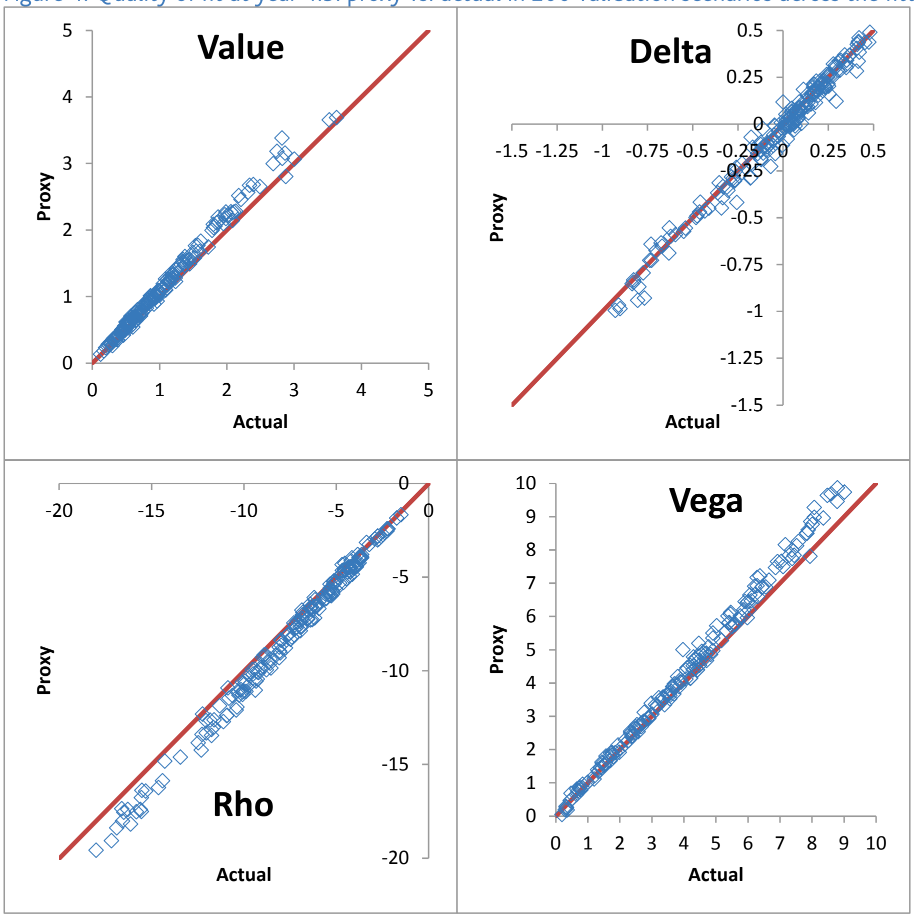


Figures 1-3 show the quality of fit to value, Delta, Rho and Vega of the lookback put option. As can be seen from Figure 1, proxy functions perform extremely well in estimating both, value and Greeks at year one. At years 5 and 9 the quality of fits deteriorates slightly, however proxy functions still provide reliable estimates. The deteriorating quality of fit can be explained by the increasing non-linearity (complexity) of the true functions that are being estimated as the option approaches maturity.

Interpolation in time

According to the fitting methodology described above, the number of fitting scenarios scales with the number of time steps at which we want to fit proxy functions. However, in order to project a hedge portfolio we need to be able to evaluate Greeks at every time where the portfolio is rebalanced, which may be as frequently as daily. In order to avoid creation of fitting scenarios at each daily step, which would be computationally infeasible, we can fit less frequently (e.g. at annual steps) and used an interpolation to calculated value and Greeks at intermediate times. As an example, below we present the proxy validation at year 4.5 using linear interpolation proxy function results between years 4 and 5.

Figure 4: Quality of fit at year 4.5: proxy vs. actual in 200 validation scenarios across the fitting space



5. Summary & Conclusions

This short note has extended the case study presented in the earlier paper on multi-period fitting of Greeks by considering the case of a more complex path-dependent form of option – a lookback option. We have found that very similar levels of quality of fit can be achieved for the Greeks of this type of option throughout its run-off, despite their greater complexity.

However, some adjustments to the fitting strategy are necessary, and this may entail a greater total scenario budget for the fitting process. This will be an ongoing area of further research.

References

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