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## Multi-Period Stochastic Scenario Generation

**Robust models are currently being developed worldwide to meet the demands of dynamic stress testing. This article describes how to build consistent projections for standard credit risk metrics and mark-to-market parameters simultaneously within a single, unified environment: stochastic dynamic macro models. It gives a step-by-step breakdown of the development of a dynamic framework for stochastic scenario generation that allows risk managers and economists to build multi-period environments, integrating conditional credit and market risk modeling.**

### Introduction

Dynamic stress testing and multi-period credit portfolio analysis are priority areas for risk managers and academics. New methodologies and techniques are being developed across the globe, mainly focusing on building robust models that translate macro scenarios into conditional risk parameters (so-called satellite models). But a significant challenge emerges when it comes to building stochastic multi-period environments. Dynamic simulations can quickly get out of control when a modeler starts increasing the sources of uncertainty and the out-of-sample periods.

In this article, we develop an innovative framework to handle multi-period stochastic simulations. The proposed methodology hinges on macro models as the starting point in the scenario generation process. Once we obtain the simulations from the econometric model, we need to embed these paths with a probability structure. To this end, we develop a rank-ordering mechanism that considers several dimensions of economic performance to produce an overall score for each scenario. With the scenarios and their probabilities in hand, we can run these forecasts through stress testing satellite models. This step provides us with forward-looking, multi-period, scenario-specific simulations for all relevant risk parameters. We illustrate this process with two leading examples: default risk for a lending portfolio (US mortgages) and mark-to-market risk for a traded credit portfolio (rate and credit spread risks).

The structure of the article is consistent with the steps required to build the proposed framework. It starts with the econometrics needed to build dynamic stochastic macro simulations (Step 1). Next, it describes our methodology for embedding the forecasted paths with probability metrics (Step 2). Satellite models are then used to compute path-specific forecasts for credit and market risk parameters (Step 3).

The combination of conditional risk parameter realizations and their probabilities

provides the modeler with the necessary inputs to address dynamic stress testing questions (such as probabilities of losses for a given stressed scenario) and to build a stochastic framework for multi-period credit portfolio management.

### Step 1: Simulations using a Dynamic Stochastic General Equilibrium (DSGE) model

To overcome the challenge of building multi-period scenarios, we propose the use of dynamic stochastic macroeconomic models. The main advantage of this family of models is the ability to simulate millions of time-series of economic shocks that are linked to each other through general equilibrium conditions. In other words, the simulations are consistent within periods (alternative macro series must satisfy equilibrium conditions), and the connection across subsequent periods comes from inter-temporal optimal behavior and pricing. This is the reason why these types of models are usually referred to as “macroeconomic models with micro-foundations.”

At the core of their set-up are optimality and arbitrage-free pricing conditions. The following sub-steps can be used to produce millions of simulated stochastic paths.

#### Step 1.A: Find the equilibrium conditions that solve the selected DSGE model

In practice, this requires the modeler to solve dynamic stochastic optimization problems. Recursive methods are leveraged in order to obtain the so-called “Bellman Equations.” These non-linear formulas represent the intra-period optimal transition for key economic variables. They provide the underlying dynamics of the system, connecting endogenous macro and financial variables with the sources of uncertainty: the shocks to economic agents and macroeconomic policies. Within these equations we also obtain the “arbitrage-free” conditions for any financial assets that are priced in the model. In other words, the equilibrium system requires market-consistent pricing for all assets over time.

**Figure 1.** Example of a standard DSGE model set-up

Workhorse Example -- Three types of agents: households, firms, government

$$\begin{aligned}
 &\circ \text{Households} \quad \max_{c_t, a_{t+1}, b_{t+1}, m_{t+1}, inv_t} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\
 &\quad s.t. \quad p_t(1 - \tau_t^{vat})c_t + m_{t+1} + p_t^a a_{t+1} + p_t^b b_{t+1} + inv_t = \\
 &\quad \quad (1 - \tau_t^w)w_t l_t + (p_t^a + p_t r_t^a)a_t + (p_t^b + i_t)b_t + m_t + r_t^k k_t \\
 &\circ \text{Firms} \quad \max_{k_t, l_t} \Pi_t = p_t f(z_t, k_t, l_t) - w_t l_t - (1 + r_t^k)k_t \\
 &\circ \text{Government} \quad \max_{T_t, B_{t+1}} \text{Welfare} \\
 &\quad s.t. \quad \tau_t + B_{t+1} = (1 + i_t)B_t + G_t \\
 &\quad \quad \tau_t = \tau_t^{vat} c_t + \tau_t^w w_t l_t + \tau_t^\pi \Pi_t
 \end{aligned}$$

Source: Moody's Analytics

**Figure 2.** Example of arbitrage-free pricing equations and other optimality conditions

Stochastic dynamic equations: “equilibrium” conditions

- **Fixed Income Arbitrage-Free** 
$$p_t^b = \beta E \left[ \frac{(p_{t+1}^b + i_{t+1})}{1 + \pi_{t,t+1}} \frac{u_c(c_{t+1}, l_{t+1})}{u_c(c_t, l_t)} * \phi(\tau_t^{vat}, \tau_t^w) \right]$$
- **Real Assets Arbitrage-Free** 
$$p_t^a = \beta E \left[ (p_{t+1}^a + p_{t+1} r_{t+1}^a) \frac{u_c(c_{t+1}, l_{t+1})}{u_c(c_t, l_t)} * \phi(\tau_t^{vat}, \tau_t^w) \right]$$
- **Relative prices** 
$$w_t = p_t f_l(z_t, k_t, l_t)$$
  

$$r_t^k = p_t f_k(z_t, k_t, l_t)$$
- **Market clearing (Supply=Demand)** 
$$f(z_t, k_t, l_t) = c_t + inv_t + G_t$$
  

$$inv_t = k_{t+1} - (1 - \delta)k_t = a_{t+1} + b_{t+1}$$
  

$$b_{t+1} = B_{t+1}$$
  

$$a_{t+1} = A_{t+1} = (1 + g_a)A_t$$
  

$$m_{t+1} = M_{t+1} = (1 + g_m)M_t$$

Source: Moody's Analytics

**Step 1.B: Build the system of stochastic differential equations that represent solutions to the model**

This step is achieved by log-linearization of equilibrium conditions around “steady-state” (the long-term solution for an economic series that is constant over time). The original macro variables get replaced by distances to the long-term values, and the non-linear system gets replaced by its first-order Taylor approximation.<sup>1</sup> This linear system is mapped into a state-space matrix form to facilitate its estimation.

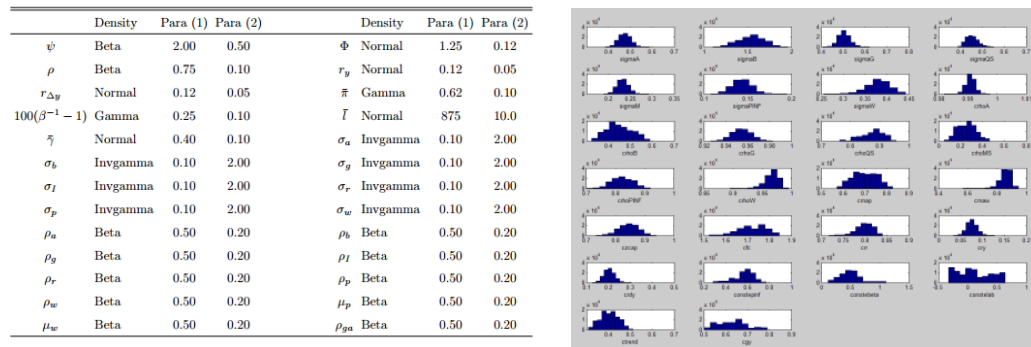
**Step 1.C: Estimate the linear system of stochastic differential equations**

Several techniques are available to estimate (or calibrate) the stochastic system of equations. There is vast and detailed literature on how to estimate DSGE models using Bayesian techniques. Fernández-Villaverde (2010) provides an overview of existing techniques and illustrates the practical advantages of Bayesian methods in the context of dynamic stochastic differential equations.<sup>2</sup>

One of the main advantages of using these methods is that after the estimation is completed the modeler has access to (posterior) distributions for all relevant parameters (“betas” that link macro series with each other and key stats for distributional assumptions of the stochastic shocks). These objects provide a very useful and rich platform for robust, forward-looking, dynamic, and consistent simulations of all endogenous variables.

<sup>1</sup> Other techniques consider higher-order approximations in order to understand the effects of non-linear relationships. See Schmitt-Grohé and Uribe (2004) for quadratic approximation methods: Schmitt-Grohé S. and Uribe M., *Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function*, Journal of Economic Dynamics & Control 28, 755–775, 2004.

<sup>2</sup> Fernández-Villaverde, J., *The Econometrics of DSGE Models*, SERIES, 1:3–49, 2010.

**Figure 3.** Bayesian estimation: prior assumptions (left) and posterior distributions for key parameters (right)

Source: Moody's Analytics

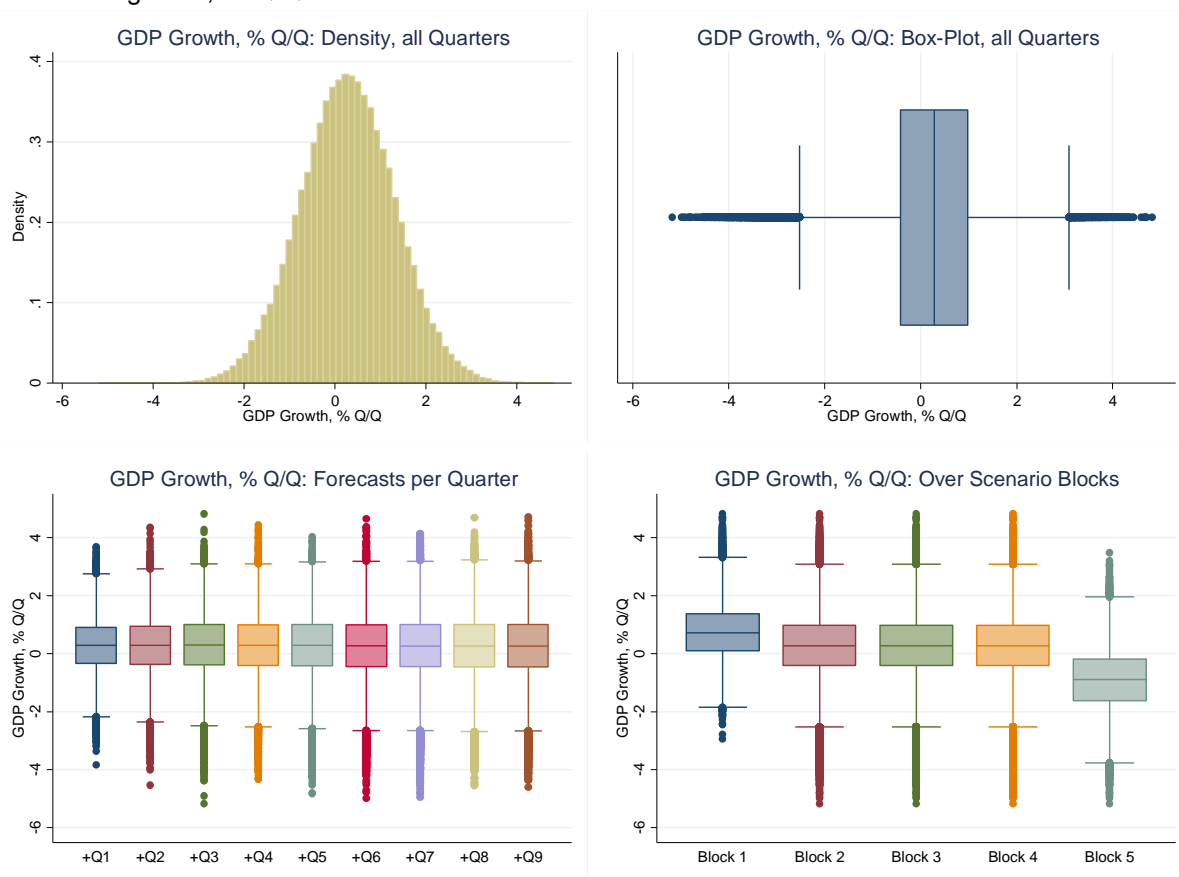
**Step 1.D: Use the estimated system to produce simulations for macro and financial series**

This critical step involves shocking the system to produce dynamic simulations out of sample. There are two sources of uncertainty that need to be considered: (a) shocks to original random variables in the model (e.g., policy surprises, productivity gains/losses, shocks to consumer preferences, etc.) and (b) the fact that estimated parameters are random variables (“coefficient uncertainty”). Statistical properties of the estimated parameters are derived from their posterior distributions. Here is where Bayesian methods have an advantage. Obtaining the posterior distributions allows a modeler to draw simulated values not only for residuals and shocks, but also for betas and other parameters.

We estimate a workhorse DSGE model for the US economy in line with Smets and Wouters (2007).<sup>3</sup> Throughout this exercise, we focus on nine consecutive quarters out-of-sample for the forecasting period (consistent with stress testing CCAR practices). These periods are labeled +Q1, +Q2, ..., +Q9. (Note that this is simply our choice for this paper and not a preference over an approach that considers longer time-horizons.)

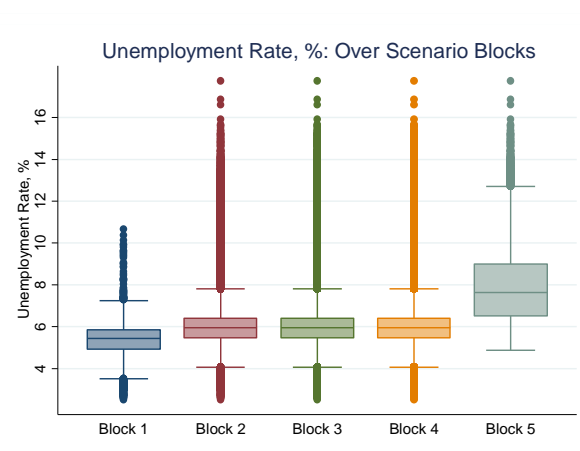
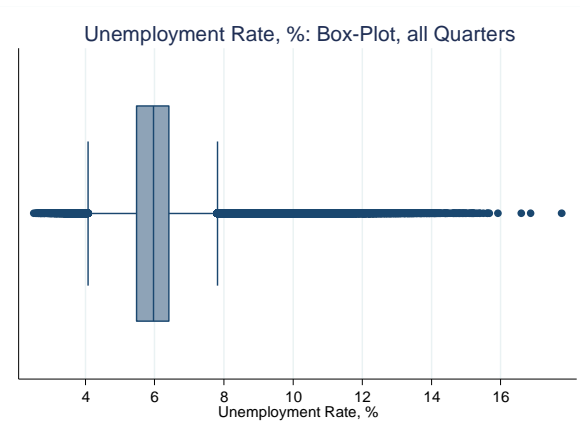
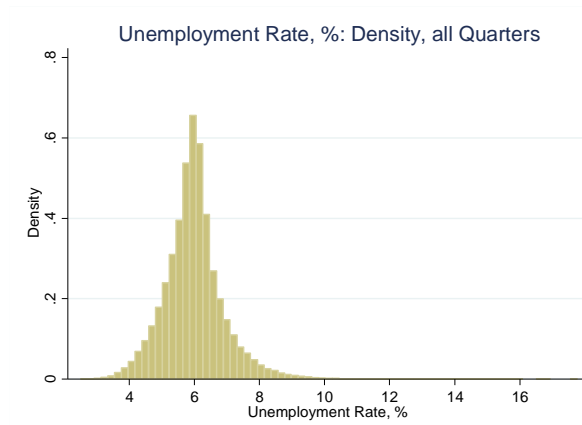
Statistical properties of simulated macro series are illustrated in Figure 4. We include GDP growth, unemployment rate, and home price dynamics as leading indicators. But DSGE models produce between 20 and 30+ economic and financial series, depending on the specifics of the version of the model selected. Histogram (frequency densities) and box-plots help a modeler understand distribution properties of the simulated paths. Some charts contain blocks of simulations as x-axis categories. These blocks represent groups of scenarios according to their severity (see section 2 for a detailed explanation on how the scenarios get rank-ordered). Block 1 groups the most optimistic forecasts while Block 5 contains stressed scenarios.

<sup>3</sup> Smets, F. and Wouters R., *Shocks and Frictions in US Business Cycles*, European Central Bank Working Paper Series N 722, 2007.

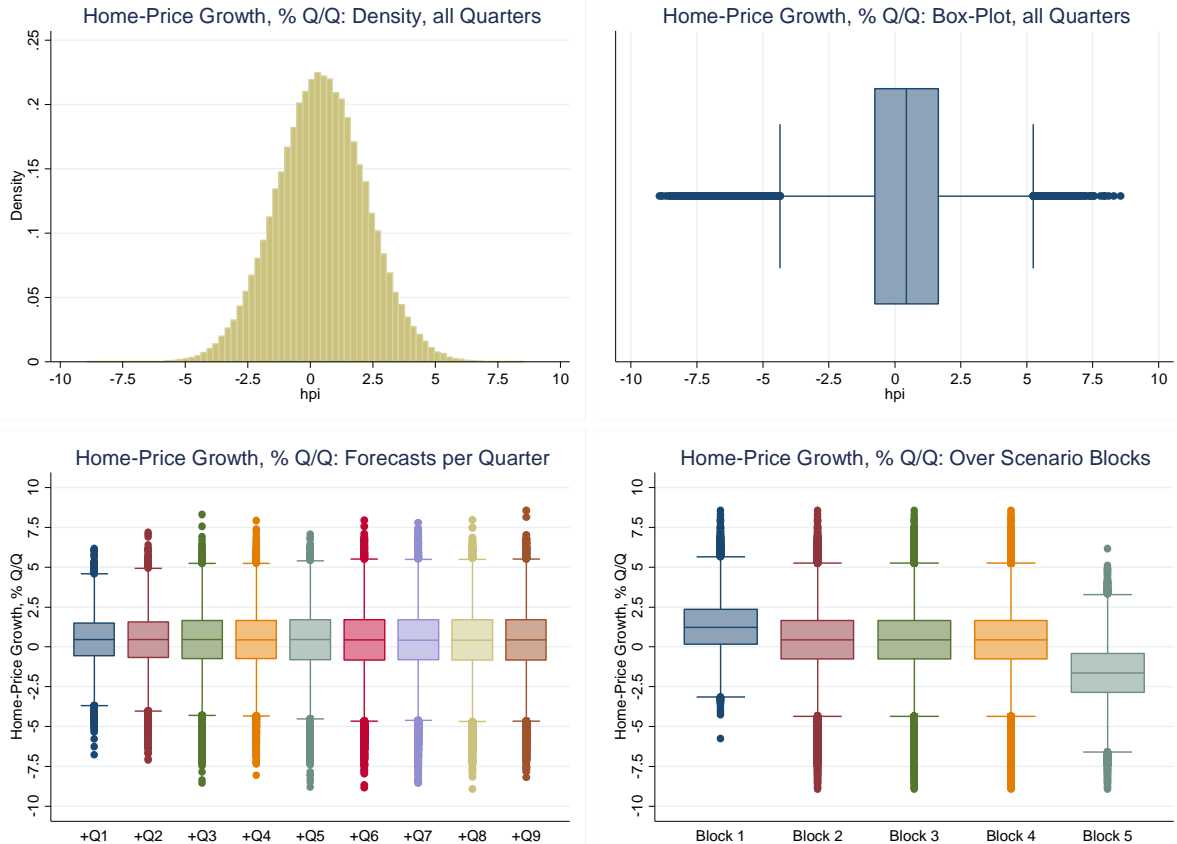
**Figure 4.** Statistical properties for key economic factors**Figure 4.1.** GDP growth, % Q/Q

Source: Moody's Analytics

**Figure 4.2.** Unemployment rate, %



Source: Moody's Analytics

**Figure 4.3.** Home price growth, % Q/Q

Source: Moody's Analytics

The result is a full set of dynamic, stochastic, and forward-looking paths for macro and financial series. The equations used to derive these simulations rest on general equilibrium conditions, making the projections consistent across variables and over time. The forward-looking attribute rests on the fact that these scenarios will vary over the business cycle. The DSGE model gets re-estimated with new data and the forecasts are conditional on the starting point of the out-of-sample period.

For notation purposes, let's refer to a given scenario as "z," wherein the vector Z contains the whole list of macro and financial series with values at all quarters-out-of-sample (+Q1 to +Q9).

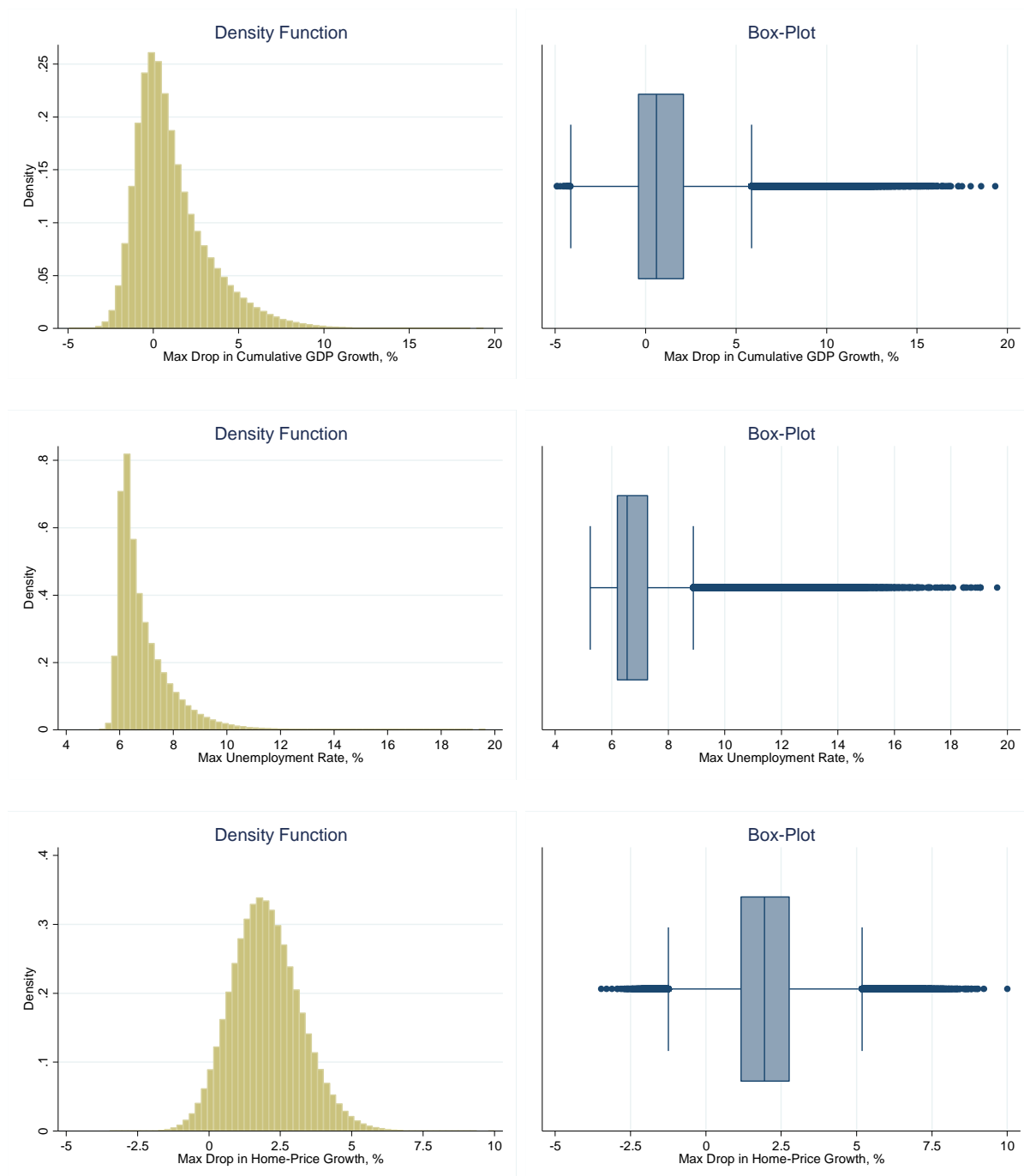
## Step 2: Embedding the stochastic scenarios with a probability structure

A natural next step is to derive probabilities and severities for the simulated scenarios. To achieve this goal, we develop a multi-factor rank-ordering mechanism that attributes severity according to 25+ dimensions of economic severity. The core macro variables that are part of the calculation are: GDP growth, unemployment rate, home price changes, consumption dynamics, investment profiles, interest rate movements, and inflation. For each of these series, we rank the scenarios according to several criteria: average and/or cumulative values over the scenarios, maximum/minimum targets, and volatilities (sigma vs. average).

The algorithm produces a combined score that translates into a ranking for each scenario. The embedded statistical structure can be obtained by (a) calculating the severity of any scenario from the percentage of the simulations that score lower or higher (purely a rank ordering exercise) or (b) grouping scenarios based on fixed intervals for score values. All scenarios within a given cluster have the same probability, given by the relative size of the interval (number of scenarios in the cluster divided by the total number of simulations).

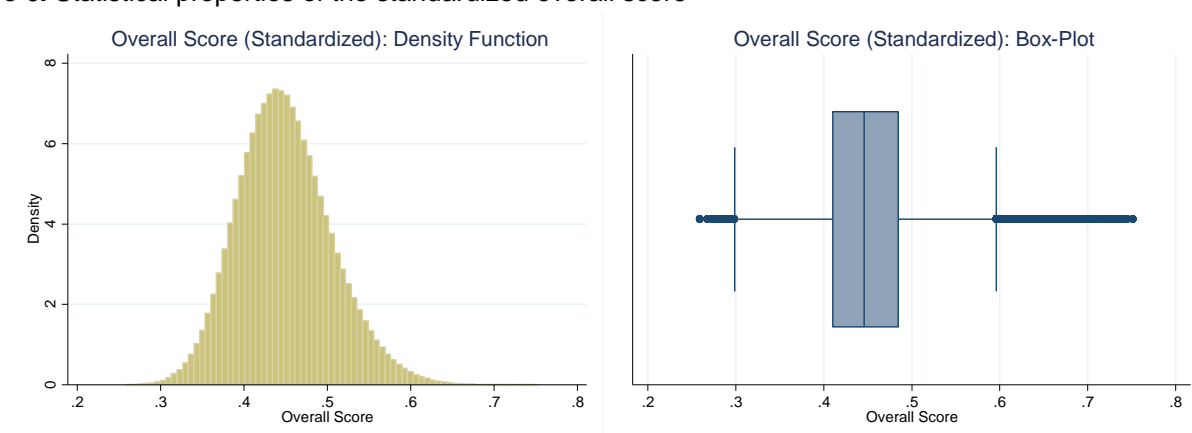
We illustrate the rank-ordering process with the properties of three marginal loadings and show the distribution properties of the final, overall score.

**Figure 5.** Leading marginal scores: minimum GDP cumulative growth rates, maximum unemployment rate, maximum drop on home price growth



Source: Moody's Analytics



**Figure 6.** Statistical properties of the standardized overall score

Source: Moody's Analytics

Step 2 provides us with a vector “ $p(z)$ ” that has a single probability value per scenario. Note that it is not time-dependent, as the scoring algorithm has considered information across all relevant macro series observed at all points in time. In other words, our stochastic shocks are represented as dynamic paths for a group of macro and financial series. Each path has an associated probability value  $p(z)$ .

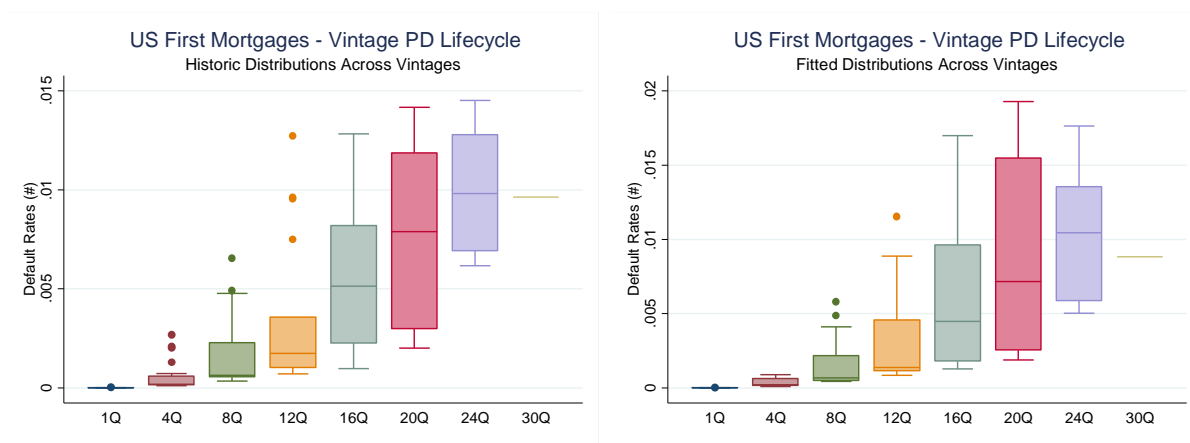
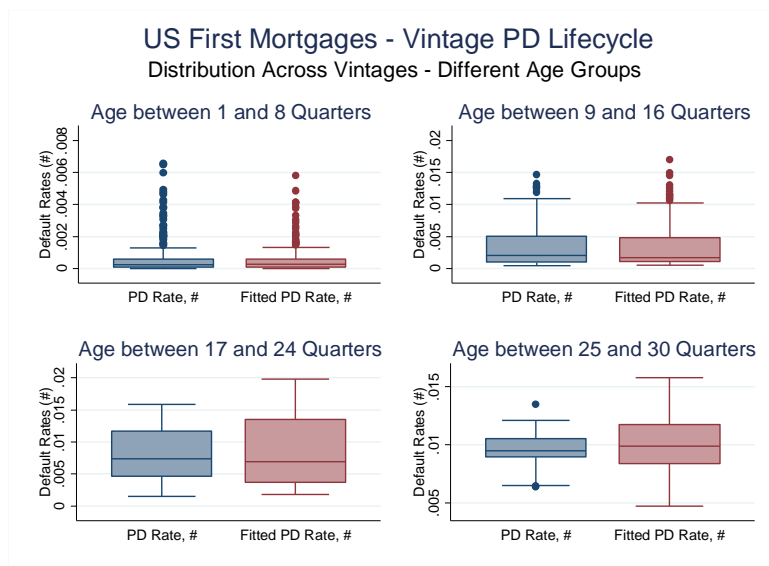
### Step 3: Connecting scenarios to risk parameters using stress testing satellite models

The last step of the process consists of linking scenarios with credit and market risk parameters. The modeler can now leverage recent developments on stress testing methodologies. The financial industry has produced a vast literature on robust models that are able to calculate risk parameters conditional on any given macro scenario. Instead of simply running a handful of multi-period scenarios, we can run thousands of them through stress testing models and obtain conditional realizations for credit metrics.

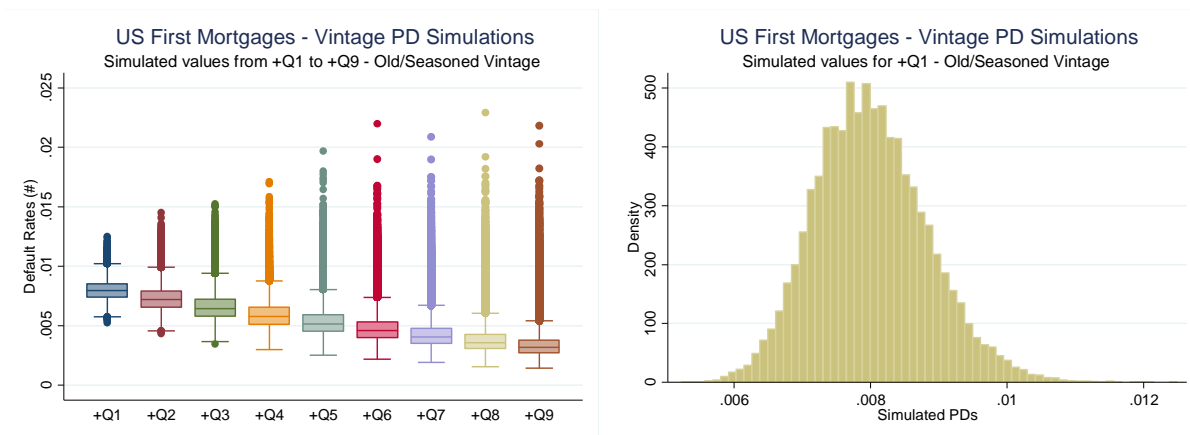
#### *Simulations of credit risk parameters: US mortgage portfolio as a leading example*

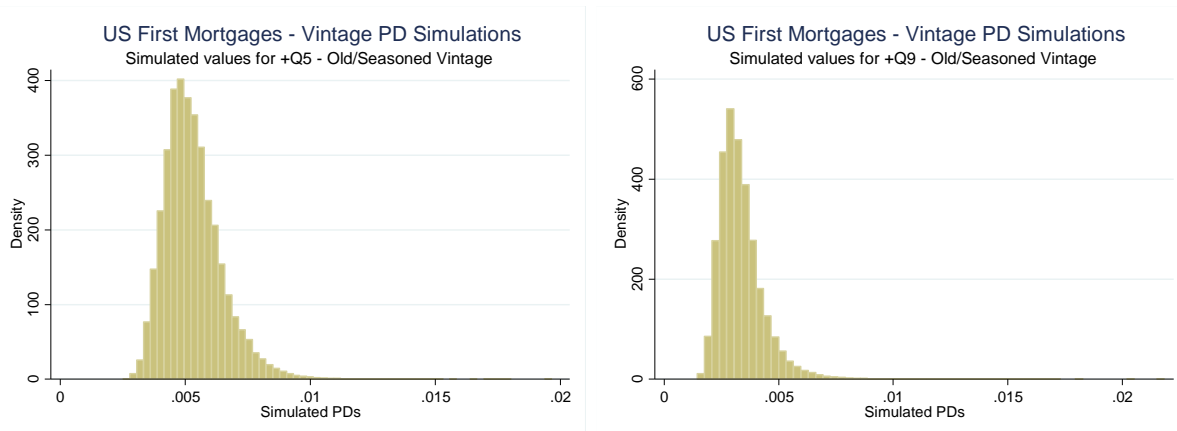
Following the methodology described in Licari and Suárez-Lledó (2013), we leverage a vintage-PD model for US first mortgages to run through the rank-ordered macro simulations.<sup>4</sup> The result is a set of dynamic paths for vintage PDs from +Q1 to +Q9 (results are illustrated in Figures 7 and 8). We need to emphasize that these PDs are not forecasted only for existing vintages, but also for loans in vintages that will be originated in the future, together with the level of forecasted volumes for these new originations. This is of particular importance when performing dynamic stress testing and multi-period portfolio credit analysis. Figure 8.2 presents the simulated PD values for a vintage of mortgages that gets originated in the first out-of-sample period (+Q1).

<sup>4</sup> Licari, J. and Suárez-Lledó, J., *Stress Testing of Retail Credit Portfolios*, Risk Perspectives Magazine, 2013.

**Figure 7.** Historic vs. fitted PDs over age intervals – PD lifecycle (term-structure)

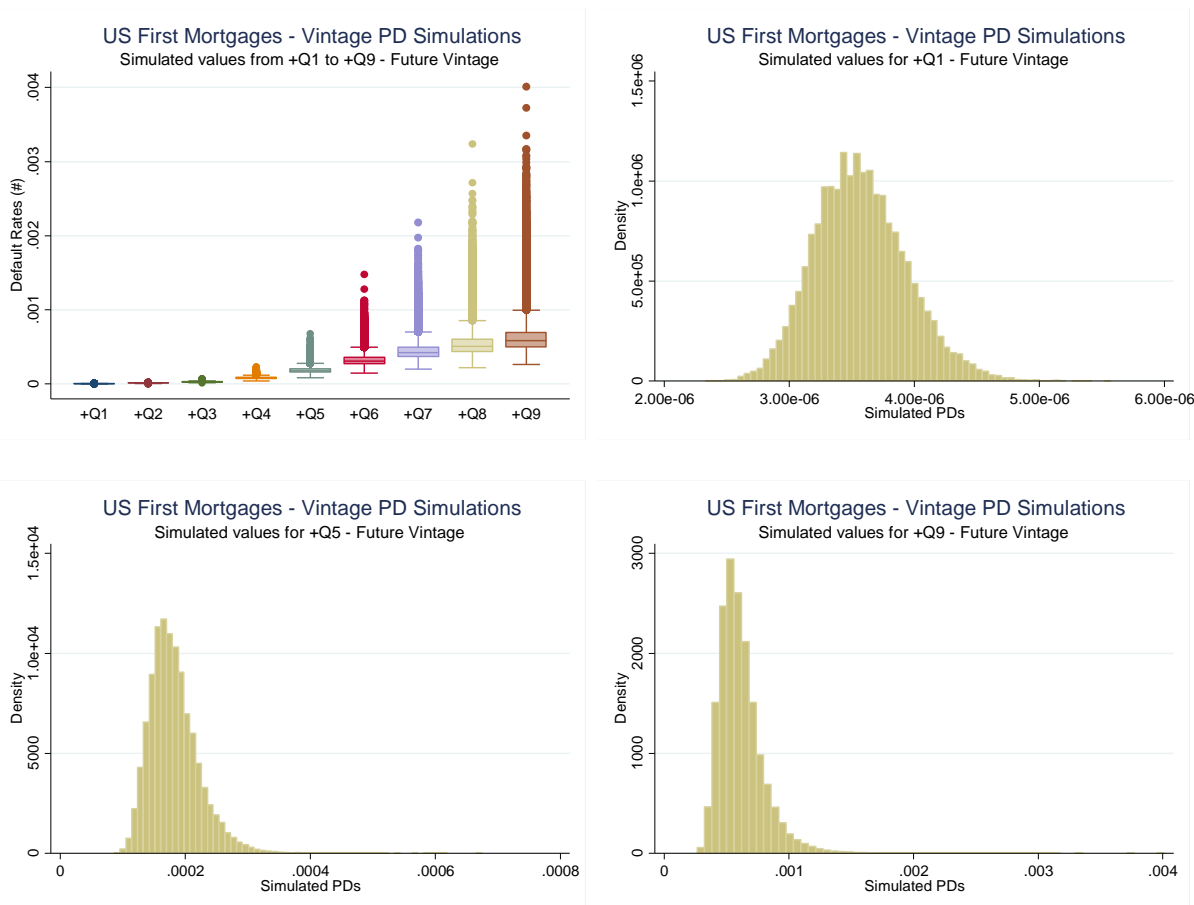
Source: Moody's Analytics

**Figure 8.** Simulated PDs across quarters-out-of-sample (+Q1 to +Q9)**Figure 8.1.** Old/seasoned vintage



Source: Moody's Analytics

**Figure 8.2.** Future vintage (booked in the out-of-sample period – dynamic projection)



Source: Moody's Analytics

The outputs of this exercise consist of simulated paths of conditional PDs (for each vintage in the mortgage portfolio and across all out-of-sample periods). These conditional PDs – combined with the probability vector  $p(z)$  – become the necessary inputs for running multi-period credit portfolio analysis.

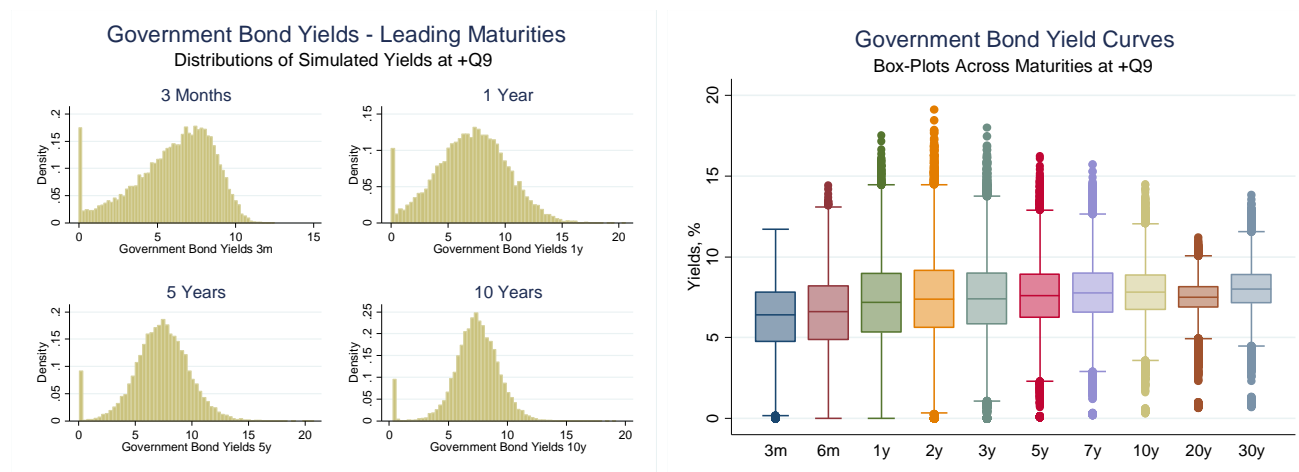
### Simulations of market risk parameters: interest rates and credit spreads as leading example

We now study the translation of the stochastic scenarios into relevant mark-to-market metrics. Government bond yields and corporate credit spreads are presented as leading examples, following econometric techniques developed in Licari, Loiseau-Aslanidi, and Suárez-Lledó (2013).<sup>5</sup> The modeling methods rest on a combination of principal component analysis and time-series estimation techniques.

It is worth highlighting the dynamic behavior of simulated term-premiums (as a proxy for the yield curve slope), as illustrated in Figure 9.3. The simulations produce different shapes, including inverted curves (negative premiums) and severe scenarios with high values for the yield-curve slope.

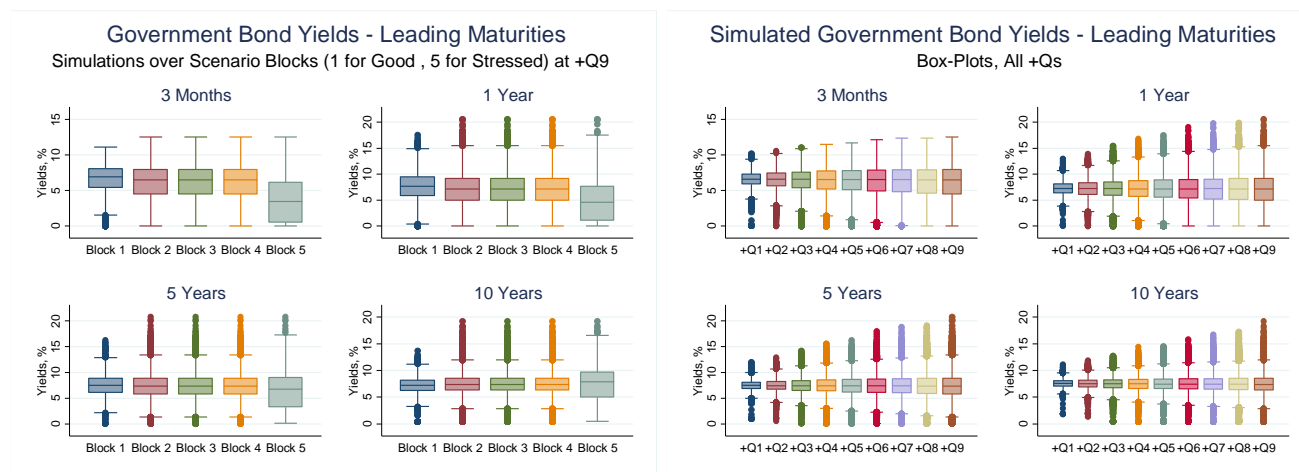
**Figure 9.** Simulated government bond yield curves

**Figure 9.1.** Distributions of yields at +Q9 for different maturity points

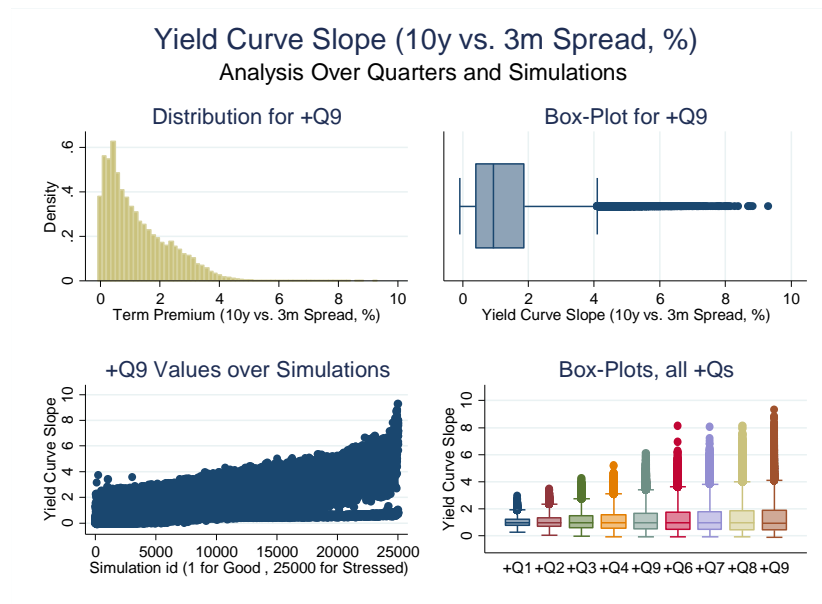


Source: Moody's Analytics

**Figure 9.2.** Yields over blocks of scenarios at +Q9 and over quarters-out-of-sample (+Q1 to +Q9)

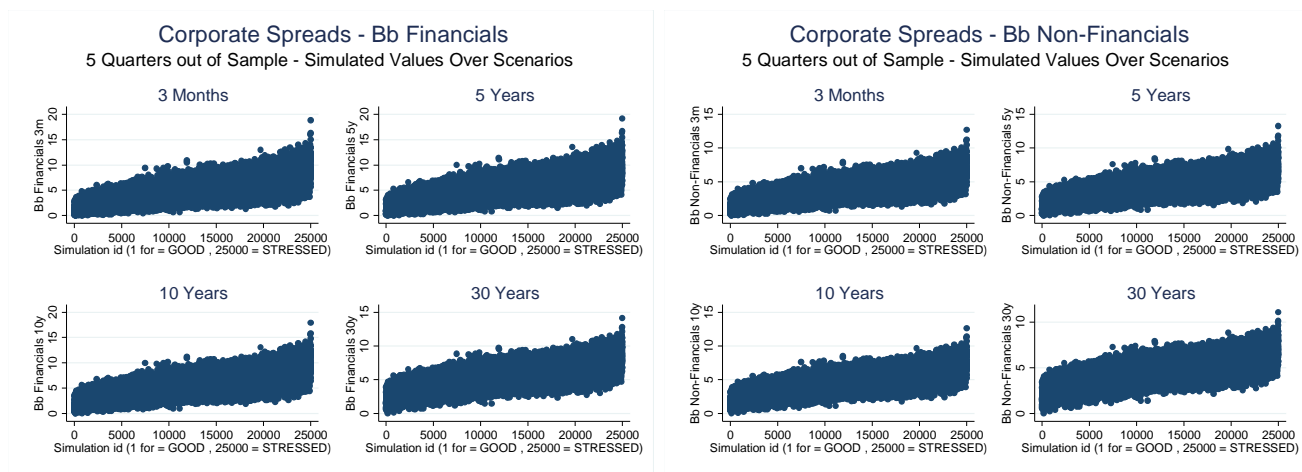


<sup>5</sup> Licari, J., Loiseau-Aslanidi, O. and Suárez-Lledó, J., *Modelling and Stressing the Interest Rates Swap Curve*, Moody's Analytics Working Paper, 2013.

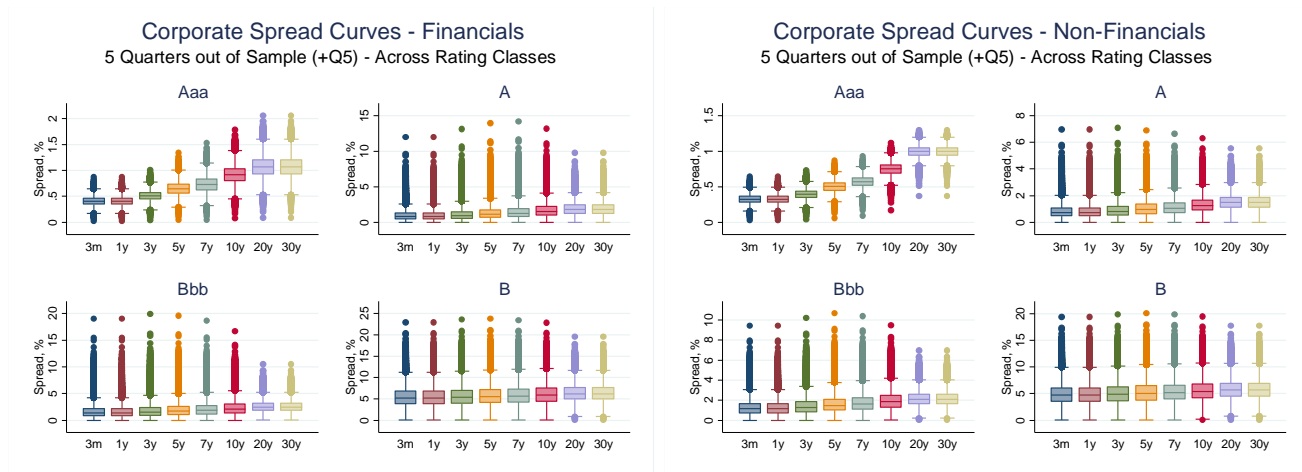
**Figure 9.3.** Term-premium spreads (yield curve slope) at +Q9

Source: Moody's Analytics

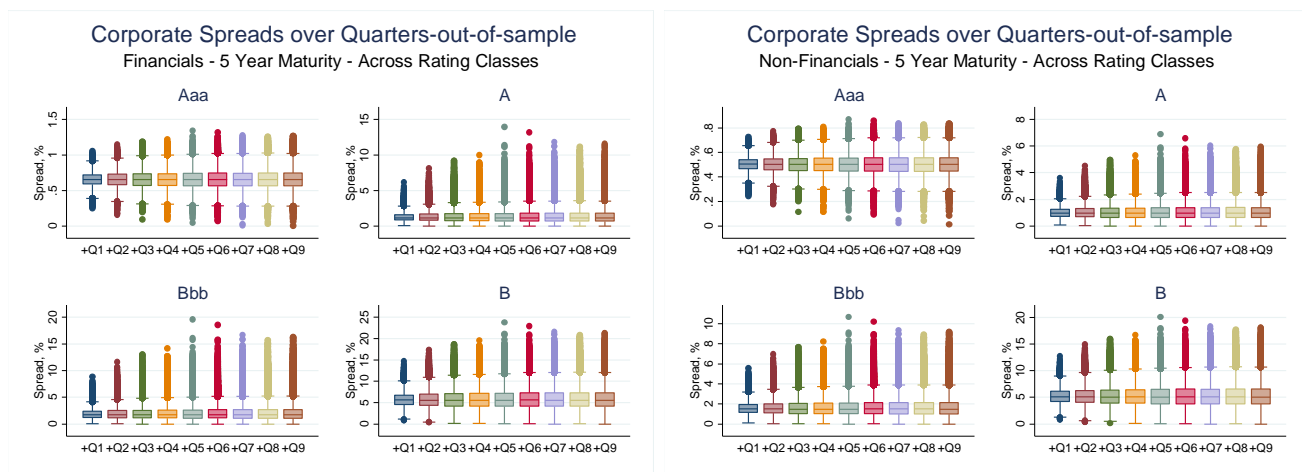
Corporate credit spreads are observed across financial and non-financial sectors. Within each sector, there is further segmentation across rating classes (Aaa, Aa, A, Bbb, Bb, and B) and maturities (3m, 1y, 3y, 5y, 7y, 10y, 20y, and 30y). Statistical properties for simulated spreads are illustrated in Figure 10.

**Figure 10.** Simulated corporate credit spreads – financials and non-financials – over rating classes and maturities**Figure 10.1.** Scatters of simulated spreads over scenarios at +Q5

Source: Moody's Analytics

**Figure 10.2.** Box plots of simulated spread curves at +Q5

Source: Moody's Analytics

**Figure 10.3.** Box plots of simulated spreads, 5-year maturity, +Q1 to +Q9

Source: Moody's Analytics

The methods described in this section provide a modeler with forward-looking, dynamic simulations for risk-free rates and credit spreads. These consistent projections together with their probabilities,  $p(z)$ , represent the building blocks for mark-to-market risk assessments for traded credit portfolios.

### Concluding remarks

In this article, we develop a dynamic framework for stochastic scenario generation. The proposed methodology sets the necessary inputs for dynamic stress testing and multi-period credit portfolio analysis.

Of particular relevance is the ability to build (simultaneously) consistent projections for standard credit risk metrics and mark-to-market parameters within a single, unified environment. In other words, using stochastic dynamic macro models as the scenario foundation allows us to integrate conditional credit and market risk modeling.

## Sources

- <sup>1</sup> Other techniques consider higher-order approximations in order to understand the effects of non-linear relationships. See Schmitt-Grohé and Uribe (2004) for quadratic approximation methods: Schmitt-Grohé S. and Uribe M., *Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function*, Journal of Economic Dynamics & Control 28, 755–775, 2004.
- <sup>2</sup> Fernández-Villaverde, J., *The Econometrics of DSGE Models*, SERIES, 1:3–49, 2010.
- <sup>3</sup> Smets, F. and Wouters R., *Shocks and Frictions in US Business Cycles*, European Central Bank Working Paper Series N 722, 2007.
- <sup>4</sup> Licari, J. and Suárez-Lledó, J., *Stress Testing of Retail Credit Portfolios*, Risk Perspectives Magazine, 2013.
- <sup>5</sup> Licari, J., Loiseau-Aslanidi, O. and Suárez-Lledó, J., *Modelling and Stressing the Interest Rates Swap Curve*, Moody's Analytics Working Paper, 2013.

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